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ON THE SYMBOLIC REPRESENTATION OF QUOTIENTIAL COEFFICIENTS OF THE SECOND ORDER.*

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The quotiential coefficient of a function of x, $y=f(x)$, has been defined by the expression†

$$\lim_{k=1} \log_k \frac{f(kx)}{f(x)}. \tag{1}$$

If we denote this limit by qy/qx, it may be shown that

$$\frac{qy}{qx}=\frac{x}{y}\cdot\frac{dy}{dx}, \tag{2}$$

and, for our present purpose, we may look upon (2) as the defining equation of the quotiential coefficient‡ of a function of x.

With the aid of this equation we derive immediately the following table of quotientation formulae:

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*Read before the San Francisco Section of the American Mathematical Society.

†*American Journal of Mathematics*, Vol. XXIV, p. 265.

‡The term quotiential coefficient originates from the expression (1) which defines it as the limit of the logarithm of a quotient just as the differential coefficient is defined as the limit of a quotient of differences.

$$\frac{qa}{qx}=0,\ a\neq 0;\ \frac{qa}{qx}=\frac{0}{0},\ a=0; \tag{3_1}$$

$$\frac{qax^n}{qx}=n,\ \frac{qx}{qx}=1; \tag{3_2}$$

$$\frac{qa^x}{qx}=x\log a; \tag{3_3}$$

$$q\frac{\log x}{qx}=\frac{1}{\log x}. \tag{3_4}$$

If $y=u\pm v\pm w\pm$ etc., u, v, w, ..., being functions of x,

$$y\frac{qy}{qx}=u\frac{qu}{qx}\pm v\frac{qv}{qx}\pm w\frac{qw}{qx}\pm \text{ etc.} \tag{3_5}$$

If $y=u\times\!\div v\times\!\div w\times\!\div$ etc.,[*]

$$\frac{qy}{qx}=\frac{qu}{qx}\pm\frac{qv}{qx}\pm\frac{qw}{qx}\pm \text{ etc.} \tag{3_6}$$

$$\frac{qu^n}{qx}=n\frac{qu}{qx}. \tag{3_7}$$

If $u=f(z)$ and $z=\phi(x)$,

$$\frac{qu}{qx}=\frac{qu}{qz}\cdot\frac{qz}{qx}, \tag{3_8}$$

$$\frac{qu}{qx}=1\Big/\frac{qx}{qu}. \tag{3_9}$$

If y is a function of n dependent variables, $y=F(u_1,\ u_2,\ \ldots)$, where $u_1=\phi_1(x)$, $u_2=\phi_2(x)$, etc.

$$\frac{qy}{qx}=\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}+\left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx}+\text{ etc.}, \tag{4}$$

the brackets being used in the Eulerian sense to denote partial variation of the dependent variables, *i. e.*,

$$\left(\frac{qy}{qu_1}\right)=\frac{u_1}{y}\frac{\partial y}{\partial u_1},\ \text{etc.}$$

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[*] ×÷ is used for the multiplication sign, ×, with dot in each of the vertical angles.

The proof of (4) is as follows:

$$\frac{dy}{dx}=\frac{\partial y}{\partial u_1}\frac{du_1}{dx}+\frac{\partial y}{\partial u_2}\frac{du_2}{dx}+\text{ etc.},$$

hence,

$$\frac{qy}{qx}=\frac{x}{y}\frac{dy}{dx}=\frac{u_1}{y}\frac{\partial y}{\partial u_1}\cdot\frac{x}{u_1}\frac{dy}{dx}+\frac{u_2}{y}\frac{\partial y}{\partial u_2}\cdot\frac{x}{u_2}\frac{dy}{dx}+\text{ etc.}$$

$$=\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}+\left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx}+\text{ etc.}$$

gy/gx is called the total quotiential coefficient of y with respect to x, and (qy/qu_1) the partial quotiential coefficient of y with respect to x.

Of course, all the preceding formulae may also be derived without a knowledge of differentiation by applying equation (1).

A partial quotiential coefficient of a partial quotiential coefficient we call a second partial quotiential coefficient, in symbols

$$\left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right]=\frac{q^2y}{qu_2qu_1}.$$

But unlike differentiation, partial quotientiation is not commutative, so that the order of the variables must be carefully attended to. We shall follow the order observed by American writers on differentiation, namely,

$$\left(\frac{q^2y}{qu_2\,qu_1}\right)=\left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right].$$

THEOREM. *If y is a function of two variables, u and v, then*

$$\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)=\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\,qu}\right). \tag{5}$$

Proof. By definition,

$$\left(\frac{q^2y}{qu\,qv}\right)=\frac{u}{\left(\frac{qy}{qv}\right)}\cdot\frac{\partial}{\partial u}\left(\frac{v}{y}\cdot\frac{\partial y}{\partial v}\right)=u\left(\frac{\partial^2y}{\partial u\,\partial v}-\frac{1}{y}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial y}{\partial u}\right)/\frac{\partial y}{\partial v},$$

$$\left(\frac{q^2y}{qv\,qu}\right)=\frac{v}{\left(\frac{qy}{qu}\right)}\cdot\frac{\partial}{\partial v}\left(\frac{u}{y}\cdot\frac{\partial y}{\partial u}\right)=v\left(\frac{\partial^2y}{\partial v\,\partial u}-\frac{1}{y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v}\right)/\frac{\partial y}{\partial u},$$

whence,

$$\left(\frac{q^2y}{qu\,qv}\right):\left(\frac{q^2y}{qv\,qu}\right)=\frac{u}{y}\cdot\frac{\partial y}{\partial u}:\frac{v}{y}\cdot\frac{\partial y}{\partial v}=\left(\frac{qy}{qu}\right):\left(\frac{qy}{qv}\right)$$

which states the theorem to be proved in the form of a proportion.

Let us next quotientiate, with respect to x, according to our rules (3_1)—(3_9), each side of the expression

$$\frac{qy}{qx}=\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}.$$

After clearing of fractions there results

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^{\,2}}{qx}+\left[\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\,qu}\right)+\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu\,qv}\right)\right]\frac{qu}{qx}\frac{qv}{qx}$$
$$+\left(\frac{qy}{qv}\right)\frac{q^2u}{qv^2}\Big)\frac{\overline{qv}^{\,2}}{qx}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\frac{q^2v}{qx^2}. \tag{6}$$

By applying (5) the expression within the brackets reduces to $2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)$ The resulting expression bears a striking resemblance to the corresponding expression for the differential quotient of the second order.

The first three terms on the right are now

$$\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^{\,2}}{qx}+2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)\frac{qu}{qx}\frac{qv}{qx}+\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)\frac{\overline{qv}^{\,2}}{qx},$$

which, by dropping the partial quotiential coefficients of the first order and replacing at the same time q^2y by $\overline{qy}^{\,2}$, assumes the form of a perfect square, namely,

$$\left(\frac{qy}{qu}\right)^2\frac{\overline{qu}^{\,2}}{qx}+2\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)\frac{qu}{qx}\frac{qv}{qx}+\left(\frac{qy}{qv}\right)^2\frac{\overline{qv}^{\,2}}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^2,$$

so that (6) may be written symbolically,

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^{(2)}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\frac{q^2v}{qx^2} \tag{7}$$

where the exponent in parenthesis signifies, that the expression to which it is attached is to be squared, and after squaring, $\left(\frac{qy}{qu}\right)^2$ is to be replaced by $\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)$, $\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)$, and $\left(\frac{qy}{qv}\right)^2$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)$.

When $u=ax^m$, $v=bx^n$, formula (7) reduces to a remarkably simple from. For in that case $\frac{qu}{qx}=m$, $\frac{qv}{qx}=n$, $\frac{q^2u}{qx^2}=0$, $\frac{q^2v}{qx^2}=0$, hence all terms on the right of (7) outside of the bracket vanish, and we have

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)m+\left(\frac{qy}{qv}\right)n\right]^{(2)}. \tag{8}$$

Formulas (6), (7), and (8) may be at once extended to functions of any number of dependent variables. Let the the dependent variables be u_1, u_2, ..., u_n, then from (4),

$$\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}.$$

Quotientiating again and remembering that generally

$$\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_k\,qu_i}\right)=\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right),\quad i,\ k=1,\ \ldots,\ n,$$

we obtain

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_i{}^2}\right)\frac{\overline{qu}_{i}{}^{2}}{qx}+2\sum_{\substack{k=2\\ i<k}}^{k=n}\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right)\frac{qu_i}{qx}\frac{qu_k}{qx}$$

$$+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}, \tag{9}$$

or in symbolic representation,

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\right]^{(2)}+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}. \tag{10}$$

If, moreover, each of the functions is of the form $u_i=a_ix^{m_i}$, the second sum in expression (10) vanishes, $qu_i/qx=m_i$, and the expression reduces to

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}m_i\left(\frac{qy}{qu_i}\right)\right]^{(2)}. \tag{11}$$